Spatial distribution of chaotic transients in unidirectional synchronisation

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Abstract

We consider the transient time to synchronisation in unidirectionally coupled dynamical systems, both discrete maps and continuous flows. We note that while in certain systems the time distribution is typical of chaotic transients, in other cases the distribution possesses a specific spatial organization.

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1 Introduction

In recent years considerable scientific effort has been devoted to the study of transient phenomena which take place during the convergence of a dynamical system towards a stable attractor. In such cases the distribution P of transient time τ has been found to scale according to an exponential law, $P(\tau) \propto e^{-\tau/\langle\tau\rangle}$, where $\langle\tau\rangle$ is the characteristic transient lifetime. If a chaotic attractor collides at the parameter value, $p = p_c$ say, with an unstable periodic orbit, then a *boundary crises* is said have occurred [1]. For $p > p_c$, the chaotic attractor is replaced by chaotic transients of length τ, with the scaling of $\langle\tau\rangle$ found to

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behave as $\langle \tau \rangle \propto (p - p_c)^{-\gamma}$, close to but just beyond the bifurcation point [1,2].

In this paper we study dissipative dynamical systems whose phase spaces possess an invariant subset $\mathbf{S}$; and specifically, when the motion in $\mathbf{S}$ is chaotic, *i.e.* the *synchronisation* of chaotic systems. A huge amount of literature exists on the subject, so here we refer the reader only to the seminal articles [3–5], and the more comprehensive listing in [6]. With engineering applications in mind for this type of cooperative motion in coupled oscillators, typically the goal is to develop robust forms of coupling which not only guarantee synchronisation but also minimise the delay time to achieve it. Of course there are many factors which influence the convergence. Different forms of coupling will introduce different rates of dissipation transverse to $\mathbf{S}$; indeed, adaptive methods of synchronisation consist of special couplings utilised to speed up, as much as possible, the convergence process [7]. However, we restrict ourselves here to unidirectional coupling between two subsystems without feedback.

An interesting point to consider is whether the scaling properties of chaotic transients hold in case of synchronisation of the two subsystems, when the resulting stable motion is chaotic. We specifically consider the question of how the transient time to synchronisation changes when we vary the rate of transverse dissipation (*i.e.* via a coupling parameter) and the starting conditions of the coupled dynamical systems. For different systems we show that delays can occur in the synchronisation process which have a specific, but not clearly defined spatial pattern in initial condition space.

2 Coupled Maps

As an illustrative example, we consider a system consisting of two coupled logistic maps in the form

$$\begin{cases} x_{n+1} = f(x_n) - K(f(x_n) - f(y_n)) \\ y_{n+1} = f(y_n) \end{cases} \tag{1}$$

where $f(x) = ax(1-x)$ and K is a real non-negative parameter. The y system can be thought of as a driving for the slave x system through the coupling term. This type of coupling produces an extended system for which the invariant set $\{\mathbf{S} : x_n = y_n, \forall n\}$ exists, *i.e.* synchronised motion. Using the value $a = 3.9$, $f(x)$ performs a chaotic motion with a maximum Lyapunov exponent numerically evaluated as $\Lambda = 0.4960$, with a standard deviation of 1.6×10^{-4}. The value of the Lyapunov exponent is determined by the average value found over a sample of 100 random initial conditions, with 5×10^6 iterations for each initial condition. An analytic expression of the threshold of stability of the invariant subset $\mathbf{S}$ (*i.e.* the synchronised state) in (1) is given by

$$K_{thr} = 1 - e^{-\Lambda}, \tag{2}$$

defined as the point in K space in which the transverse Lyapunov exponent $\lambda_\perp$ is zero, that defines the *blowout bifurcation* [8], in this case $K_{thr} = 0.3911 \pm 0.0003$. Interestingly, the numerically evaluated threshold of synchronisation does not exactly coincide with the semi-analytical estimation from (2). We summarise the situation in Fig. 1 indicating a confidence interval for the estimation of K_{thr}. In numerical simulations synchronisation was achieved for $K < K_{thr}$, down to a value $K_1 \simeq 0.380$ for the whole phase space to be synchronised, and as low as $K_0 \simeq 0.370$ for at least a few initial points to become synchronised. Nevertheless, these two values of the coupling parameter are located outside the interval of confidence for K_{thr} given by the error distribution of Λ.

Non-trivial basins exist for the convergence of the full system onto $\mathbf{S}$, in the range $K \in [K_0, K_1]$. The dynamics in the range $K \in [K_0, K_{thr}]$ show an on-off intermittent transient whether they eventually synchronise or not. Computations were initially carried out in absence of noise, but a similar discrepancy between the numerical and the semi-analytical value of K_{thr} is also detected in presence of noise of amplitude 10^{-10}. Some further details and results investigating this discrepancy are discussed elsewhere [9].

To investigate the distribution of transient times to synchronisation in this coupled map system (1), we introduce the simplification of taking a section for a fixed x_0 and checking the transient distribution times of all the initial conditions y_0. This is a reasonable approach to take since the dynamics transverse to $\mathbf{S}$ is dependent mainly upon the dynamics of the driving system. We note that basins of attraction for synchronisation within a cutoff time in the (x_0, y_0) space consist of horizontal lines in the y_0 variable.

We examine the changes in the form of the distribution $P(n)$ of transients to convergence towards $\mathbf{S}$ when the value of K is altered. In Figure 2 three distributions $P(n)$, for K close to K_0, are given for the transient time to synchronisation of the system of coupled logistic maps in a linear-log scale. The linear fit in the scale reveals that the three distributions $P(n)$ follow exponential laws $P(n) \propto e^{-n/\langle n \rangle}$, as is the case for chaotic transients after a boundary crisis. The value for $\langle n \rangle$, for each value of K, is obtained by a log-log plot of $\langle n \rangle$ versus $K - K_0$, where $K_0 \simeq 0.370$ is the first value of K at which we note convergence onto $\mathbf{S}$. If we assume a relation of the form $\langle n \rangle \propto (K - K_0)^{-\gamma}$, a power law relation fits the data, with the exponent evaluated as $\gamma \simeq 5.31$. We have performed the same analysis with another map on the unit interval, the *skew tent map*, $f(x) = \frac{x}{c}$ for $x \in [0, c]$, and $f(x) = \frac{1-x}{1-c}$, for $x \in [c, 1]$ for $c = 0.307$, finding a similar value for γ.

For a boundary crisis, a power law scaling with approximately $\gamma = \frac{1}{2}$ can be derived by looking at the collision of an unstable periodic orbit with the basin of the attractor prior to collision [2]. However, in the present case, we are dealing with an uncountable infinity of periodic orbits whose degree of transverse stability modulates the decay mode onto $\mathbf{S}$. This scaling law applies close to the blowout bifurcation point, where the probablility distribution $P(n)$ has an exponentially decaying form. For larger values of $K - K_0$, $P(n)$ gradually changes its shape, from an exponential profile to one more Gaussian-like. This transition may be justified if we consider that as K is increased, more and more unstable periodic orbits gain transverse stability, and the transient on-off intermittent behaviour turns into an almost regular decay, whose decay rate is provided by the conditional Lyapunov exponent $\lambda_\perp$. In this way the Gaussian distribution is already generated by the time a trajectory, once $|x_n - y_n| \simeq 10^{-15} - 10^{-16}$, stays at that level of roundoff error before suddenly jumping onto $\mathbf{S}$.

As a clarifying example, consider the logistic map at fully-developed chaos, $a = 4$. For this value of a no discrepancy was found between the analytically predicted (K_{thr}) and numerically computed (K_0) threshold of stability in $\mathbf{S}$ at $K = \frac{1}{2}$. For $a = 4$ the logistic map is known to be conjugate to the symmetric tent map, and effectively for both systems coupled as in (1), the profile of the distribution $P(n)$ is always Gaussian-like, never exponentially decaying, no matter how close K is to $\frac{1}{2}$. On the contrary, for the asymmetric tent map, the profile of $P(n)$ for varied K is qualitatively the same as portrayed in Figure 3. In the case of the logistic maps, one could be tempted to relate the difference to the smoothness of the natural invariant density (a similar argument has been used in [10] to explain the different scalings of on-off intermittent behaviour), but we note that for a symmetric tent map *all solutions*, whether stable or unstable, periodic or chaotic, change transverse stability at $K = \frac{1}{2}$, so there are no intermediate processes in further increasing K once the threshold has been passed.

For the case of $a = 4$, in which the natural invariant density is smooth, an analytical estimate of the position of the peak of the Gaussian distribution $P(n)$ can be obtained. It is known that for such systems $K_{thr} = 1 - e^{-\Lambda}$, where $\lambda_\perp = \log(1 - K) + \Lambda$, and if $d_n = |x_n - y_n|$, the decay of a single trajectory towards the synchronised state is characterized by $d_n \propto e^{\lambda_\perp n}$. Numerically a limiting cutoff value, say C, has to be determined, to decide whether a trajectory is deemed to have become synchronised, and we take this have occurred after n_p iterations, hence as $d_{n_p} = C$. Thus $n_p \simeq C/\lambda_\perp$, and from the previous expressions,

$$n_p \simeq \frac{\log C}{\log \frac{1-K}{1-K_{thr}}}. \tag{3}$$

With the value of C set at 10^{-15}, approximately the roundoff error of the computer, and $K_{thr} = \frac{1}{2}$, we obtain a reasonable agreement with the numerical data, shown in Figure 4, but in fact $C = 10^{-13}$ is probably the value which gives the best correspondence, as also shown on the figure.

Returning to the case of $a = 3.9$, the arrangement of the unstable periodic orbits embedded in **S** can give rise to a particular spatial structure that aggregates points converging to the invariant subset with a certain time delay. Our interpretation is that a transversely unstable periodic orbit in a zone of K values for which the whole phase space is the basin of **S**, creates a neighbourhood of itself (and its pre-images) in which synchronisation is delayed, because trajectories passing through this neighbourhood experience a transverse repulsion. Certainly, this does not mean that this neighbourhood will be the last to converge onto **S**, but it will certainly not be the first.

To support this statement we show in Figure 5 a representation in the full 2D phase space xy of the trajectories whose convergence towards the synchronised state is the slowest. The coupled logistic maps system has been set to the parameters $a = 3.9$ and $K = 0.450$, and the points shown are the trajectories whose convergence times are in the interval $470 < n < 600$. This value of K is chosen because, even though quite far from the blowout bifurcation, it is located very close to the change of transverse stability of the unstable period-2 (now transversely stable) and the zero fixed point (still transversely unstable). This upper bound ($n = 600$) represents the tail of the distribution, so these points are the slowest to achieve synchronisation. For $a = 3.9$, the chaotic motion is stable in **S**, but the two fixed points, $y = 1 - \frac{1}{a} = 0.743589744$, and $y = 0$, are both transversely unstable. The largest accumulation of strips is actually viewed in a neighbourhood of the non-zero fixed point, giving us the impression that its transverse instability is a source of delay for the convergence onto **S** of all trajectories starting closeby.

3 Coupled oscillators

We move now to the case of coupled chaotic flows and consider the motion of the Duffing oscillator [11]. In this case another particular spatial distribution exists for the transient motion of trajectories before converging towards the synchronised state, but this time we argue a different reason for the delay observed.

The numerical experiment consists of two identical Duffing oscillators coupled together, with a linear unidirectional coupling type to yield:

$$\begin{cases} \ddot{x} + \mu\dot{x} - x + x^3 + K(x-y) = A\cos(\omega t) \\ \ddot{y} + \mu\dot{y} - y + y^3 = A\cos(\omega t), \end{cases} \tag{4}$$

where $\mu = 0.1$, $A = 3.0$, and $\omega = 0.3$. Previous numerical experiments carried out in [12] reveal that the threshold of stability of synchronised motion is located in the interval $[2.0, 2.1]$. With a fixed value of the coupling parameter, $K = 3.0$, the distribution of the transient time to synchronisation possesses a particular spatial distribution in the space of the initial conditions of the driving system $(y_0, \dot{y}_0)$. As stated in [12], the transient behaviour of the response system before exponentially converging onto **S** does not show on-off intermittency, but it is rather a chaotic evolution with a larger amplitude than the synchronised state. In the $(x, \dot{x})$ space this large scale motion forms a stable attractor, in an interval of $K < K_{thr}$ in the neighbouhood of the threshold, leaving us the impression of a chaotic solution replaced, after a crisis, by a chaotic transient. Figure 6 has been derived fixing the initial condition for the response system to be $(x_0, \dot{x}_0) = (1.1, -0.5)$, a typical value yielding chaotic dynamics, and varying the initial conditions of the driving y system with the notation $y_0 = x_0 + \xi_0$, $\dot{y}_0 = \dot{x}_0 + \eta_0$. In this notation (ξ_0, η_0) is the deviation in the initial conditions space from the invariant subset. We set a grid of initial starts in the (ξ_0, η_0) space and we record for each point on the grid the time needed to converge (within a pre-set limit) onto **S**. All initial conditions checked oscilllate around the chaotic attractor of the driving system with a large scale orbit for τ_0 cycles of the periodic forcing before starting their exponentially decay towards the synchronised state. For each trajectory time was run forwards until the Euclidean distance $d(t)$ between $(x(t), \dot{x}(t))$ and $(y(t), \dot{y}(t))$ reaches a cutoff value of 10^{-6}. Thereafter, the decay of the Euclidean distance is of the form $d(t) \propto \exp(\lambda_\perp t)$, where $\lambda_\perp$ is the maximum transverse Lyapunov exponent. As the computed $\lambda_\perp$ yields the same value for almost all initial conditions in the basin of attraction of the synchronised state [8], the exponential decay of $d(t)$, after the τ_0 cycles of orbiting transient, is approximatively the same for all initial conditions checked.

In Figure 6 we plot the spatial distribution of initial conditions for the driving system (ξ_0, η_0) whose orbiting time τ_o is less than 6 cycles of the periodic forcing, *i.e.* those points which rapidly converge. A histogram of distribution for all initial conditions in the range $(\xi_0, \eta_0) \in [-8, 8] \times [-8, 8]$ has a shape very similar of the one for $K = 0.425$ in Figure 3 for the coupled logistic maps, while the cutoff $\tau_o < 6$ places the section of all initial conditions forming the part of the distribution before half the maximum of the peak. Figure 6 has been derived with a grid of 201×201=40401 initial conditions in $(\xi_0, \eta_0) \in [-4, 4] \times [-4, 4]$.

We observe that this peculiar transient behaviour before the convergence takes place, is not necessarily observed at other parameters of the Duffing oscillator considered, but it is structurally stable in an interval of parameter values considered. The same qualitative behaviour has also been observed in cases when the dynamics in the invariant subset is periodic, inside a periodic window in a bifurcation diagram, while for a periodic case at the beginning of a bifurcation diagram, no particular spatial structure has been noted. We note that a section (ξ_0, η_0) of those initial conditions which take longest time to synchronise also displays a complex spatial structure, with a similar organized pattern of Figure 6.

4 Conclusions

In this paper we have discussed the distribution of transient time to synchronisation in typical unidirectionally coupled chaotic dynamical systems, using both discrete and continuous systems. In the cases considered a delay in the time to synchronisation was noted when compared with semi-analytic predictions.

In the coupled maps system the distribution of transient times resembles the distribution of chaotic transients, even though the evaluated exponent of the power-law scaling in parameter space is quite different ($\gamma \simeq 5.3$ compared to $\gamma = \frac{1}{2}$). We stress, however, that we have no *apriori* reasons why the time to synchronisation should follow a power law scaling as in the case of chaotic transients. The derivation of this scaling law was subjected to some intrinsic limitations. First of all, only 40 points were used to produce a fit. Values of $K < K_1$ were not considered because not all points converge, also values of K too far from the threshold (either numerical of theoretical) were not used because for larger values of K, $P(n)$ gradually changes its shape, so the fit is representative only of a limited region of the K space. In this case the numerical experiments were carried out in a parameter range reasonably close to the blowout bifurcation, so that the mutation of the distribution $P(n)$ in changing the parameter K is due to the number of unstable periodic orbits that gradually change transverse stability. Increasing the coupling parameter increases the amount of transverse dissipation, at the same time making more and more unstable periodic orbits become transversely stable. The straightforward result is a faster decay of typical trajectories onto $\mathbf{S}$, but the phase space can be locally formed of zones with a transverse repelling action, as the example reported in Figure 5.

Unstable saddle orbits can also cause a delay when they are located outside $\mathbf{S}$ when trajectories approach their stable manifolds. Another possible cause could be the existence of a "trapping region" following a bifurcation, so that trajectories can wander for quite a long time within a region of the phase space, tracing out a path seemingly of a previously existing chaotic attractor, but then eventually converging onto another stable solution, as in the case of chaotic transients after a boundary crisis [2]. This description resembles the situation encountered in the dynamics of the coupled Duffing systems. In the continuous case, in which a driven Duffing oscillator models the dynamics in the invariant subset **S**, there is a particular spatial pattern in the initial-conditions space. During the transient time recorded, the response system performs a chaotic wandering around the driving system, but with much larger amplitude before decaying exponentially onto the synchronised state. Before the blowout bifurcation this large amplitude motion was found to be stable, resulting in a trapping region for the response system trajectories after the threshold.

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Fig. 1. Schematic representation of events as the coupling parameter K is increased.

Fig. 2. Probability distributions for the system of coupled logistic maps for three values of K. The linear-log scale helps to show that the average distribution is exponential with a decay rate $-1/\langle n\rangle$.

Fig. 3. Four plots of the distribution $P(n)$ versus n for the system of coupled logistic maps with $a = 3.9$ and for four different values of K. For $K \gtrsim K_{thr}$ (top left), close to the threshold of transverse stability of $\mathbf{S}$, the distribution shows the expected exponential scaling while for increasing coupling strength the distribution appears with a peak shifted gradually towards the right. Note that the y axes are different.

Fig. 4. Position of the abscissa n_p of the peak of the distribution of synchronisation times for the coupled logistic maps at fully developed chaos ($a = 4$). The open circles represent the numerical points while the two solid lines represent the function (3) for the values $C = 10^{-15}$, and $C = 10^{-13}$, as indicated by the arrows.

Fig. 5. A case of spatial distribution of the initial conditions whose evolution is the slowest towards the synchronised state for the coupled logistic map system, for $a = 3.9$, and $K = 0.450$. More specifically, the data represent all initial conditions for which synchronisation between the drive and response variables takes place in the interval $470 < n < 600$.

Fig. 6. Spatial distribution for $\tau_o < 6$ of points converging fastest onto $\mathbf{S}$. The section is drawn in the interval $[\xi_0, \eta_0] = [-4, 4] \times [-4, 4]$, to reveal its structure.